Solutions to the Quantized Knizhnik-Zamolodchikov Equation and the Bethe-Ansatz

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Abstract

We give an integral representation for solutions to the quantized Knizhnik-Zamolodchikov equation (qKZ) associated with the Lie algebra $\mathfrak{gl}_{N+1}$. Asymptotic solutions to qKZ are constructed. The leading term of an asymptotic solution is the Bethe vector – an eigenvector of the transfer-matrix of a quantum spin chain model. We show that the norm of the Bethe vector is equal to the product of the Hessian of a suitable function and an explicitly written rational function. This formula is a generalization of the Gaudin-Korepin formula for a norm of the Bethe vector. We show that, generically, the Bethe vectors form a base for the $\mathfrak{gl}_2$ case.

Introduction

The quantized Knizhnik-Zamolodchikov equation (qKZ) is a holonomic system of difference equations introduced recently in [1,2,3]. qKZ inherits many remarkable properties of the differential Knizhnik-Zamolodchikov equation (KZ). In particular, there is an integral representation for solutions to qKZ [4] associated to $\mathfrak{gl}_{N+1}$ or $U_q(\mathfrak{gl}_{N+1})$ which has a similar structure of an integrand as an integral representation for solutions to KZ [5,6]. The $N = 1$ case was considered earlier in [7,8,9]. (Cf. also [10,11,12]). Asymptotic solutions to qKZ [13] obtained from an integral representation establish a connection between the Bethe ansatz and qKZ. The $\mathfrak{gl}_{N+1}$ and $U_q(\mathfrak{gl}_{N+1})$ analogues of the Gaudin-Korepin formula for the norm of the Bethe vector can be proved up to a multiplicative constant in this framework. Similar results for KZ are obtained in [14,15].

In this paper we consider only the case of qKZ associated to $\mathfrak{gl}_{N+1}$ although almost all the results can be lifted to the $U_q(\mathfrak{gl}_{N+1})$ case, cf. [4,13]. Section 2 is based on [4] and Sections 3,4 are based on [13].

1. *qKZ* associated with $\mathfrak{gl}_{N+1}$

Let $\mathfrak{g}=\mathfrak{gl}_{N+1}$ with the canonical generators $\{E_{ij}\}$. Let Y be the corresponding Yangian with a coproduct Δ. Let $\varphi: Y\to U(\mathfrak{g})$ be the natural homomorphism and let θ_z, $z\in\mathbb{C}$, be the canonical automorphism of Y. Set $\varphi_z=\varphi\circ\theta_z$. For any two highest weight $\mathfrak{g}$-modules V_1,V_2 with generating vectors v_1,v_2, respectively, there is a unique R-matrix $R_{V_1V_2}(z)\in\mathrm{End}\,(V_1\otimes V_2)$, such that for any $X\in Y$

$$R_{V_1V_2}(z_1-z_2)\,(\varphi_{z_1}\otimes\varphi_{z_2})\circ\Delta(X)=(\varphi_{z_1}\otimes\varphi_{z_2})\circ\Delta'(X)\,R_{V_1V_2}(z_1-z_2) \qquad (1)$$

in $\mathrm{End}\,(V_1\otimes V_2)$ and $R_{V_1V_2}(z)\,v_1\otimes v_2=v_1\otimes v_2$. Here $\Delta'=P\circ\Delta$ and P is a permutation of factors in $Y\otimes Y$. $R_{V_1V_2}(z)$ preserves the weight decomposition of a $\mathfrak{g}$-module $V_1\otimes V_2$; its restriction to any weight subspace of $V_1\otimes V_2$ is a rational function in z. For any $\mu\in\mathbb{C}^{N+1}$ introduce $L(\mu)=\exp\Bigl(\sum_{i=1}^{N+1}\mu_iE_{ii}\Bigr)$, which is well defined in any highest weight $\mathfrak{g}$-module.

Let $V_1,\dots,V_n$ be highest weight $\mathfrak{g}$-modules, $V=V_1\otimes\ldots\otimes V_n$. Let $\rho_i:\mathrm{End}\,(V_i)\to\mathrm{End}\,(V)$ be embeddings as tensor factors. Set $R_{ij}(z)=\rho_i\otimes\rho_j(R_{V_iV_j}(z))$ and $L_i(\mu)=\rho_i(L(\mu))$. Let $p\in\mathbb{C}$ and $z=(z_1,\dots,z_n)$. The operators

$$\begin{aligned}K_i(z;p)=R_{i,i-1}(z_i-z_{i-1}+p)\ldots R_{i1}(z_i-z_1+p)\,\times\\ \times\,L_i(\mu)\,R^{-1}_{ni}(z_n-z_i)\ldots R^{-1}_{i+1,i}(z_{i+1}-z_i)\end{aligned} \qquad (2)$$

are called the *qKZ operators*. Denote by Z_i the p-shift operator:

$$Z_i:\Psi(z_1,\dots,z_n)\mapsto\Psi(z_1,\dots,z_i+p,\dots,z_n)\,.$$

The *quantized Knizhnik-Zamolodchikov equation* [1] is the holonomic system of difference equations for a V-valued function $\Psi(z;p)$:

$$Z_i\Psi(z;p)=K_i(z;p)\Psi(z;p)\,,\qquad i=1,\dots,n\,. \qquad (3)$$

2. Integral representations for solutions to *qKZ*

Let $t=(t_1,\dots,t_\ell)$. Let Q_a be the p-shift operator with respect to a variable t_a. Let $\Phi(t,z;p)$ be a meromorphic scalar function and $w(t,z;p)$ a V-valued rational function in t,z. Say that $\Phi(t,z;p)w(t,z;p)$ gives an *integral representation* for solutions to system (3) if

$$Z_i(\Phi w)-K_i\,\Phi w=\sum_{a=1}^{\ell}\bigl(Q_a(\Phi w_{ai})-\Phi w_{ai}\bigr)\,,\qquad i=1,\dots,n\,,$$

for suitable rational functions $w_{ai}(t,z;p)$. $\Phi(t,z;p)$ and $w(t,z;p)$ are called the *phase function* and the *weight function*, respectively.

Theorem 1. *There exists an integral representation for solutions to qKZ* (3) *associated with* $\mathfrak{gl}_{N+1}$.

We describe the integral representation for solutions to *qKZ* more explicitly below.

Fix $\lambda \in \mathbb{Z}^N_{\geqslant 0}$. Let $\Lambda(1), \dots , \Lambda(n) \in \mathbb{C}^{N+1}$ be highest weights of $\mathfrak{g}$-modules V_1, $\dots , V_n$, respectively. Let V_λ be the weight subspace:

$$V_\lambda = \bigl\{ v \in V \mid E_{ii}\, v = \bigl(\lambda_{i-1} - \lambda_i + \sum_{m=1}^{n} \Lambda_i(m) \bigr)\, v\,, \;\; i = 1, \dots , N+1 \bigr\}$$

where $\lambda_0 = \lambda_{N+1} = 0$. The *qKZ* operators preserve the weight decomposition of a $\mathfrak{g}$-module V . Now we are interested in solutions to system (3) with values in V_λ . Set $\ell = \sum_{i=1}^{N} \lambda_i$. Let $t = (t_{11}, \dots , t_{1\lambda_1}, t_{21}, \dots , t_{2\lambda_2}, \dots , t_{N1}, \dots , t_{N\lambda_N}) \in \mathbb{C}^{\,\ell}$. The phase function is given as follows:

$$\begin{aligned} \Phi(t, z; p) = {} & \prod_{m=1}^{n} \prod_{i=1}^{N+1} \exp\bigl(z_m \mu_i \Lambda_i(m)/p\bigr) \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} \exp\bigl(t_{ij}(\mu_{i+1} - \mu_i)/p\bigr) \times \\ & \times \prod_{m=1}^{n} \prod_{i=1}^{N} \prod_{j=1}^{\lambda_i} \frac{\Gamma((t_{ij} - z_m + \Lambda_i(m))/p)}{\Gamma((t_{ij} - z_m + \Lambda_{i+1}(m))/p)} \times \\ & \times \prod_{i=1}^{N} \prod_{j=2}^{\lambda_i} \prod_{k=1}^{j-1} \frac{\Gamma((t_{ik} - t_{ij} - 1)/p)}{\Gamma((t_{ik} - t_{ij} + 1)/p)} \prod_{i=1}^{N-1} \prod_{j=1}^{\lambda_i} \prod_{k=1}^{\lambda_{i+1}} \frac{\Gamma((t_{i+1,k} - t_{ij} + 1)/p)}{\Gamma((t_{i+1,k} - t_{ij})/p)}\, . \end{aligned} \tag{4}$$

The weight function $w(t, z)$ is given by an algebraic construction taken from the nested Bethe ansatz. In particular, $w(t, z)$ does not depend on p, μ at all. For more details cf. [4, 13].

3. Asymptotic solutions to *qKZ*

Let $p \to 0$. We are interested in asymptotic solutions to system (3) which have the form

$$\Psi(z; p) = \exp\bigl(\tau(z)/p\bigr) \sum_{s=0}^{\infty} \Psi_s(z)\, p^s\, . \tag{5}$$

The phase function has an asymptotic expansion:

$$\Phi(t, z; p) \simeq a(p)\, \exp\bigl(\tau(t, z)/p\bigr)\, \Xi(t, z)\, \bigl(1 + \sum_{s=1}^{\infty} \phi_s(t, z)\, p^s \bigr)$$

where $a(p)$, $\tau(t, z)$, $\Xi(t, z)$, $\phi_s(t, z)$ are suitable functions.

A point (t, z) is called a *critical point* if $\exp\Bigl(\dfrac{\partial \tau}{\partial t_a}(t, z)\Bigr) = 1$ for $a = 1, \dots , \ell$. Set $H(t, z) = \det\Bigl(\dfrac{\partial^2 \tau}{\partial t_a \partial t_b}(t, z)\Bigr)$. A critical point (t, z) is called *nondegenerate* if

$H(t,z)\neq 0$. Equations for critical points coincide with the Bethe ansatz equations in the nested Bethe ansatz. The set of critical points is preserved by the natural action of the product of the symmetric groups $\mathbf{S}=\mathbf{S}_{\lambda_1}\times\ldots\times\mathbf{S}_{\lambda_N}$ on variables t.

Let $(t^\star,z^\star)$ be a nondegenerate critical point. Set $I_a=\frac{1}{2\pi i}\frac{\partial\tau}{\partial t_a}(t^\star,z^\star)$, $I(t)=\exp\bigl(-2\pi i\sum_{a=1}^{\ell}I_a t_a/p\bigr)$ and $\hat\tau(t,z)=\tau(t,z)-2\pi i\sum_{a=1}^{\ell}I_a t_a$. Let D be a suitable small real disk containing $(t^\star,z^\star)$, $\dim_{\mathbb{R}}D=\ell$. Set

$$\Psi(z;p)=\frac{1}{a(p)}\Bigl(-\frac{1}{2\pi p}\Bigr)^{\ell/2}\int\limits_D I(t)\,\Phi(t,z;p)\,w(t,z;p)\,d^\ell t\,.$$

As $p\to 0$, by the method of steepest descend $\Psi(z;p)$ has an asymptotic expansion

$$\Psi(z;p)\simeq\exp\bigl(\hat\tau(t(z),z)/p\bigr)\,\Xi(t(z),z)\,H^{-\frac12}(t(z),z)\,\Bigl(w(t(z),z)+\sum_{s=1}^{\infty}\psi_s(t(z),z)\,p^s\Bigr)$$

where a function $t(z)$ is such that $(t(z),z)$ is a nondegenerate critical point and $t(z^\star)=t^\star$.

Theorem 2. *Let $\Phi(t,z;p)w(t,z;p)$ be an integral representation for solutions to qKZ* (3)*. The asymptotic expansion of $\Psi(z;p)$ as $p\to 0$ gives an asymptotic solution to system* (3) *of the form* (5)*.*

Lemma 3. *Let (t,z) be a nondegenerate critical point. Then*

$$K_i(z;0)w(t,z)=\exp\Bigl(\frac{\partial\tau}{\partial z_i}(t,z)\Bigr)\,w(t,z)\,,\qquad i=1,\ldots,n\,.$$

A critical point (t,z) is called an *offdiagonal* critical point if $t_{ij}\neq t_{ik}$ for $(i,j)\neq(i,k)$, and a *diagonal* critical point, otherwise.

Theorem 4. *Let $(t^\star,z^\star)$ be a diagonal nondegenerate critical point. Then* $\exp\bigl(-\hat\tau(t(z),z)/p\bigr)\,\Psi(z;p)=O(p^\infty)$ *as* $p\to 0$.

Let S_i be the Shapovalov form on V_i. Set $S=S_1\otimes\ldots\otimes S_n$. Let $K_i^\dagger(z;p)$ be the dual to $K_i(z;p)$ with respect to the form S. Set $\mathrm{R}(z)=\prod_{j=2}^{n}\prod_{i=1}^{j-1}R_{ij}(z_i-z_j)$ both indices in the ordered product increasing from the left to the right.

Lemma 5. i) $K_i^\dagger(z;p)=\mathrm{R}(z)\,Z_i\bigl(K_i(z;-p)\mathrm{R}^{-1}(z)\bigr)$.
ii) *Operators $R_{ij}(z)$ and $\mathrm{R}(z)$ are symmetric with respect to the form S.*

Set $\langle w_1,w_2\rangle=S(\mathrm{R}w_1,w_2)$. Let $(t(z,\mu),z)$ be an offdiagonal nondegenerate critical point.

Theorem 6. $\langle w(t(z,\mu),z), w(t(z,\mu),z)\rangle = \mathrm{const}\,\Xi^{-2}(t(z,\mu),z)\,H(t(z,\mu),z)$ *where* const *does not depend on continuous deformations of the critical point* $(t(z,\mu),z)$.

Conjecture. *For any offdiagonal critical point* (t,z)

$$\langle w(t,z), w(t,z)\rangle = (-1)^{\ell}\,\Xi^{-2}(t,z)\,H(t,z)\,.$$

For any critical points $(t,z)\,,(\bar t,z)$ *lying in different* $\mathbf{S}$*-orbits* $\langle w(\bar t,z), w(t,z)\rangle = 0$.

This Conjecture was proved for the $\mathfrak{gl}_2$ case in [13] using the limit $\exp(\mu_2-\mu_1)\to 0$. A combinatorial proof for the first part of Conjecture for the $\mathfrak{gl}_2$ case was given in [16], and for the $\mathfrak{gl}_3$ case (with a special choice of $\mathfrak{g}$-modules) in [17]. For similar results for the differential KZ equation cf. [14, 15].

Let $\mathfrak{C}(z,\mu)$ be the set of all offdiagonal critical points modulo the action of the group $\mathbf{S}$. Vectors $w(t,z)$ are preserved by the action of $\mathbf{S}$ modulo multiplication by a nonzero scalar factor.

Theorem 7. *Let* $\mathfrak{g}=\mathfrak{gl}_2$. *Let* $z,\ \mu,\ \Lambda(1),\dots,\Lambda(n)$ *be generic. Then* $\{w(t,z)\}_{t\in\mathfrak{C}(z,\mu)}$ *is a base in* V_λ .

4. *qKZ* and bases of singular vectors.

Assume that $\mu=0$. Set $\mathrm{Sing}\,V = \bigl\{ v\in V \mid E_{i,i+1}\,v=0\,,\ \ i=1,\dots,N \bigr\}$ and $\mathrm{Sing}\,V_\lambda = V_\lambda\cap \mathrm{Sing}\,V$.

Lemma 8. *Let* (t,z) *be an offdiagonal critical point. Then* $w(t,z)\in \mathrm{Sing}\,V_\lambda$.

Let $\mathfrak{g}=\mathfrak{gl}_2$. Let $\mathfrak{C}(z)$ be the set of all offdiagonal critical points modulo the action of the symmetric group $\mathbf{S}_\lambda$. Let $\Lambda_1(m)-\Lambda_2(m)<0\,,\ \ m=1,\dots,n$, or let $\Lambda(1),\dots,\Lambda(n)$ be generic.

Theorem 9. *For generic* z *all offdiagonal critical points are nondegenerate. Moreover,* $\#\mathfrak{C}(z)=\dim\,\mathrm{Sing}\,V_\lambda$ *and* $\{w(t,z)\}_{t\in\mathfrak{C}(z)}$ *is a base in* $\mathrm{Sing}\,V_\lambda$.

Assume that $\Lambda_1(m)-\Lambda_2(m)\in\mathbb{Z}_{\geqslant 0}\,,\ \ m=1,\dots,n$. Let $V_1,\dots,V_n$ be the irreducible $\mathfrak{g}$-module with highest weights $\Lambda(1),\dots,\Lambda(n)$, respectively. A critical point (t,z) is called a *trivial* critical point if $w(t,z)=0$, and a *nontrivial* critical point, otherwise. Let $\mathfrak{C}(z)$ be the set of all nontrivial critical points modulo the action of the symmetric group $\mathbf{S}_\lambda$.

Theorem 10. *For any* z *all offdiagonal trivial critical points are degenerate. For generic* z *all nontrivial critical points are nondegenerate. Moreover,* $\#\mathfrak{C}(z)=\dim\,\mathrm{Sing}\,V_\lambda$ *and* $\{w(t,z)\}_{t\in\mathfrak{C}(z)}$ *is a base in* $\mathrm{Sing}\,V_\lambda$.

For similar results for the differential KZ equation cf. [14, 15] .

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